From Propositional Logic

Rules of Inference

$$\alpha \vdash \beta$$

$$\frac{\alpha}{\beta}$$

Modus Ponens

$$\frac{\alpha \rightarrow \beta, \alpha}{\beta}$$

Rules of Inference

And-Elimination

$$\frac{\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n}{\alpha_i}$$

And-Introduction

$$\frac{\alpha_1, \alpha_2, \ldots, \alpha_n}{\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n}$$

Rules of Inference

Or-Introduction

$$\frac{\alpha_i}{\alpha_1 \vee \alpha_2 \ldots \vee \alpha_n}$$

Double Negation Elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

frametitleRules of Inference **Unit Resolution**

$$\frac{\alpha \vee \beta, \neg\beta}{\alpha}$$

Resolution

$$\frac{\alpha \vee \beta, \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

Inference in First-Order Logic

$SUBST(\theta, \alpha)$

$SUBST(\{x/sam, y/pam\}, likes(x, y)) likes(sam, p$

Rules of Inference

Universal Elimination

For any sentence α, v a variable, and g a ground term.

$$\frac{\forall v\ \alpha}{SUBST(\{v/g\}, \alpha)}$$

From $\forall$x likes(x, icecream), we can infer

- *likes(ben, icecream)*
- *likes(jerry, icecream)*

Rules of Inference

Existential Elimination

For any sentence α, v a variable, and k a constant symbol that does not appear anywhere else in the knowledge base:

$$\frac{\exists v\ \alpha}{SUBST(\{v/k\}, \alpha)}$$

From $\exists$x likes(x, icecream), we can infer

- *likes(oliver, icecream)*

Rules of Inference

Existential Introduction

For any sentence α, v a variable that does not occur in α, and g a ground term that does occur in α:

$$\frac{\alpha}{\exists v\ SUBST(\{g/v\}, \alpha)}$$

From likes(jerry, icecream), we can infer

- *$\exists$ x likes(x, icecream),*

First-Order Theorem Proving

Clause Form for First-Order Logic
A literal is an atomic sentence or the negation of an atomic sentence.
A clause is a set of literals representing their disjunction.
All variables are implicitly universally quantified.
$\{\text{on}(x, \text{a})\}$ represents $\forall x\ \text{on}(x, \text{a})$
$\{\neg\text{on}(x, \text{a}), \text{above}(\text{f}(x), \text{b})\}$ represents
$\forall x\ \neg\text{on}(x, \text{a}) \vee \text{above}(\text{f}(x), \text{b})$

First-Order Theorem Proving

Substitutions
A substitution is any finite set of associations between variables and expressions in which

1. each variable is associated with at most one expression and
2. no variable with an associated expression occurs within any of the associated expressions.

The terms associated with the variables in a substitution are often called bindings for these variables.
A substitution can be applied to a predicate calculus expression to produce a new expression; the substitution instance $\beta\theta$.

First-Order Theorem Proving

Substitutions
EXAMPLE

$\{x/a, y/f(b), z/w\}$

$p(x, x, y, v)$

$p(a, a, f(b), v)$

First-Order Theorem Proving

Substitutions A set of expressions $\alpha_1 \ldots \alpha_n$ are unifiable if and only if there is a substitution σ that makes the expressions identical.

First-Order Theorem Proving

Unification

EXAMPLE

$p(a, y, z)$

and

$p(x, b, z)$

are unifiable with the substitution

$\{x/a, y/b, z/c\}$

to yield

$p(a, b, c)$

First-Order Theorem Proving

Most General Unifier Consider again, the unifier of

$p(a, y, z)$

and

$p(x, b, z)$

$\{x/a, y/b, z/c\}$

$p(a, b, c)$

Other Unifiers are also Possible:

$\{x/a, y/b, z/d\}$
$\{x/a, y/b, z/f(c)\}$
$\{x/a, y/b, z/w\}$
$\vdots$

First-Order Theorem Proving

Most General Unifier But the Most General Unifier (MGU) makes the least commitment.

$\{x/a, y/b\}$

$p(a, b, z)$

UNIFY (α, β) *returns the MGU of* α *and* β.

Theorem Proving

Resolution Rule of Inference

$$\frac{\alpha \vee \beta_1, \neg\beta_2 \vee \delta}{(\alpha \vee \delta)\theta}$$

where θ = *mgu of* β_1 *and* β_2.

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Set Notation

$$\frac{\Gamma \text{ with } \beta_1 \in \Gamma \quad \Delta \text{ with } \neg\beta_2 \in \Delta}{(\Gamma - \beta_1) \cup (\Delta - \neg\beta_2)\theta}$$

Theorem Proving

The Procedure

1. CLAUSES $\leftarrow$ Clausify $(\Delta \cup \neg\alpha)$
2. Repeat
   - Pick two clauses in CLAUSES, c_1, c_j, such that c_i and c_j have a resolvent r_{ij} not already in CLAUSES.
   - If no such c_i, c_j exist then return $(\Delta \not\models \alpha)$.
   - If $r_{ij} = \Box$, then return $(\Delta \models \alpha)$.
   - Otherwize add r_{ij} to CLAUSES.

Theorem Proving

Examples

1. $\neg$c(x) $\vee$ s(x)
2. $\neg$c(x) $\vee$ r(x)
3. c(a)
4. o(a)
5. $\neg$o(x) $\vee$ $\neg$r(x)
6. r(a) 3,2
7. $\neg$r(a) 5,4
8. $\Box$ 6,7

First-Order Theorem Proving

Quantifiers

- $\neg\forall v\ \varphi \Leftrightarrow \exists v\ \neg\varphi$
- $\neg\exists v\ \varphi \Leftrightarrow \forall v\ \neg\varphi$

First-Order Theorem Proving

Elimination of Existential Quantifiers Skolemization
Existential quantifier not in scope of a universal
$\exists x$ p(x) is replaced by p(a)

Where "a" is a Skolem constant.
"a" must be a new constant symbol that does not occur anywhere else in the database.

Otherwise
$\forall x\ \forall y\ \exists z$ p(x, y, z) is replaced by $\forall x\ \forall y$ p(x, y, f(x, y))

Where "f" is a Skolem symbol.
"f" must be a new function symbol.

Conversion to Clause form

1. Eliminate Implications

$$(\forall x\, p(x)) \rightarrow (\exists y\, q(y))$$

is replaced by

$$\neg(\forall x\, p(x)) \vee (\exists y\, q(y))$$

Conversion to Clause form

2. Move Negations Inwards

$\neg((\forall x\, p(x)) \vee (\exists x\, q(x)))$

$(\neg\forall x\, p(x)) \wedge (\neg\exists x\, q(x))$

$(\exists x\, \neg p(x)) \wedge (\forall x\, \neg q(x))$

Conversion to Clause form

3. Rename Variables

$(\exists x\, p(x)) \wedge (\forall x\, q(x))$

$(\exists x\, p(x)) \wedge (\forall y\, q(y))$

Conversion to Clause form

4. Convert to Prenex Form

$(\forall x\, p(x)) \vee (\exists y\, q(y))$

$\forall x\, \exists y\, (p(x)) \vee q(y))$

Prefix: string of quantifiers.
Matrix: quantifier-free formula

Conversion to Clause form

5. Eliminate Existential Quantifiers

$(\forall y\ (\exists x\ p(x,y))$

$(\forall y\ p(g(y),y)$

where "g" is a new function symbol, a Skolem function.

Conversion to Clause form

6. Put matrix in Conjunctive Normal Form Distribute

$(\mathrm{p}(x) \wedge \mathrm{q}(x,y)) \vee \mathrm{q}(z)$ becomes
$(\mathrm{p}(x) \vee \mathrm{q}(z)) \wedge (\mathrm{q}(x,y) \vee \mathrm{q}(z))$

Conversion to Clause form

7. Eliminate Universal Quantifiers

$\forall x\ \forall y\ \mathrm{p}(x, f(x), y)$ becomes
$\mathrm{p}(x, f(x), y)$

Conversion to Clause form

8. Separate into Clauses

$\mathrm{p}(x) \wedge \mathrm{p}(y)$ becomes
$\{\mathrm{p}(x)\}$ and $\{\mathrm{p}(y)\}$

Conversion to Clause form

9. Rename Variables

Rename variables – so that no variable symbol appears in more than one clause.

$$\{p(x)\}, \{q(x)\} \text{ becomes}$$
$$\{p(x)\}, \{q(y)\}$$