Employing the Overlapping Solution FEM to Multiple Scatterers

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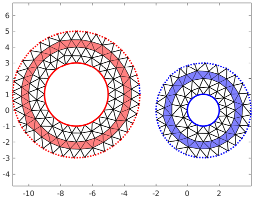
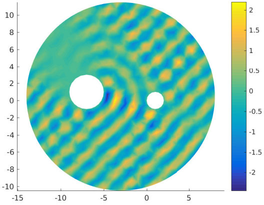




Abstract

The H^1-conforming overlapping solution FEM will be employed to compute the scattered field in the setting of multiple scatterers. In particular, the variational forms in the context of a single computational domain as well as *multiple* (disjoint) domains will be compared. This is followed by some preliminary computational results - single and iterative solves.

Outline

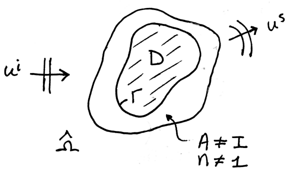
- The original problem
  - Truncate the domain (R, u^{sca}, $\mathcal{L}$)
  - Variational form - H^1 conforming
  - Numerical example with two scatterers
- New problem - Multiple Meshes
  - Modified variational form
  - Numerical example with two scatterers
  - Single solve
  - Iterative scheme
- Concluding Remarks

The Problem We Consider-Domain and Boundaries

Consider the bounded scatterer D (with smooth boundary Γ) and set $\widehat{\Omega}$ to be the unbounded complement of $\overline{D}$ in $\mathbb{R}^2$. Determine u satisfying

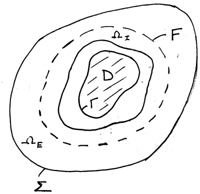
$$\begin{aligned}
\nabla \cdot \mathcal{A}\nabla u + k^2 n u &= f \qquad \text{in} \quad \widehat{\Omega}, && (1)\\
u &= 0 \qquad \text{on} \quad \Gamma, && (2)\\
\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) &= 0, && (3)\\
u &= u^i + u^s \qquad \text{in} \quad \widehat{\Omega} && (4)
\end{aligned}$$

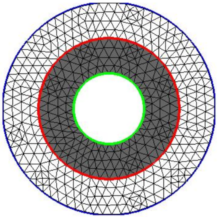
where $\mathcal{A}$ is a complex 2×2 bounded matrix and n is piecewise uniformly continuous in $\widehat{\Omega}$.



The Truncated Problem

- Let F be a closed uniformly Lipschitz curve surrounding D and Σ a closed uniformly Lipschitz curve surrounding F.
- Γ, F and Σ have no point in common.
- The curve Σ serves as the outer boundary for the new truncated domain.
- Ω is the bounded part of $\widehat{\Omega}$ inside of Σ
- Ω_I and Ω_E are the parts of Ω that are located interior and exterior to F, respectively.



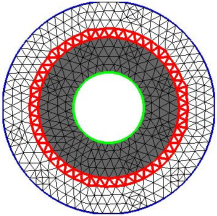


Cut-Off Function

Define the cut-off function denoted by $R(\boldsymbol{y})$ such that $R = 0$ in a neighborhood of Σ, $\mathcal{N}(\Sigma)$, and $R = 1$ in a neighborhood of F, $\mathcal{N}(F)$. That is to say, for $\boldsymbol{x} \in \Sigma$,

$$R(\boldsymbol{y})\Phi(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \Phi(\boldsymbol{x},\boldsymbol{y}) & \boldsymbol{y} \in \mathcal{N}(F), \\ \\ 0 & \boldsymbol{y} \in \mathcal{N}(\Sigma). \end{cases}$$

where Φ is the fundamental solution to the general helmholtz equation (1).



$R \neq 0$ shown in red.

Representation of Scattered Field

Represent the scattered field for $\boldsymbol{x}$ outside of F (using Green's first identity):

$$u^s(\boldsymbol{x}) = \int_{\Omega_E} \nabla_{\boldsymbol{y}} u^s(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} R(\boldsymbol{y}) \Phi(\boldsymbol{x},\boldsymbol{y})\, dA_{\boldsymbol{y}} - k^2 \int_{\Omega_E} R(\boldsymbol{y}) \Phi(\boldsymbol{x},\boldsymbol{y})\, u^s(\boldsymbol{y}) dA_{\boldsymbol{y}}$$

$$- \int_F u^s(\boldsymbol{y}) \frac{\partial \Phi}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\, ds_{\boldsymbol{y}} := I^R[u^s, \Phi](\boldsymbol{x}).$$

If we define the boundary operator on Σ as

$$\mathcal{L}(u) := \left(\frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{x}}} - i\lambda u \right)\Big|_{\Sigma}, \quad \lambda \in \mathbb{R} \setminus \{0\},$$

where $\boldsymbol{n}_{\boldsymbol{x}}$ is the outward normal to Σ, a reduced problem is to find $u \in W$ such that

$$\nabla \cdot \mathcal{A} \nabla u + k^2 n u = 0 \quad \text{in} \quad \Omega, \tag{5}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{6}$$

$$\mathcal{L}\left(u - I^R[u, \Phi]\right) = \mathcal{L}\left(u^{inc}\right) \quad \text{on} \quad \Sigma, \tag{7}$$

where

$$W := \left\{ f \in L^2(\Omega) : \ f_x, f_y \in L^2(\Omega) \text{ and } f|_{\Gamma} = 0 \right\}.$$

Variational Form H^1

The variational formulation is to determine $u \in W$ such that

$$a(u,v) = \ell(v) \qquad \forall v \in W$$

where $a(\cdot,\cdot)$ is the sesquilinear form defined on W by

$$\begin{aligned} a(u,v) \quad &= \quad \int_\Omega \overline{\nabla v} \cdot \mathcal{A} \nabla u dA - k^2 \int_\Omega \overline{v} n u dA \\ &\quad - \int_\Sigma \overline{v} \mathcal{L}\left(I^R[u,\Phi] \right) ds - i\lambda \int_\Sigma \overline{v} u ds \end{aligned}$$

and $\ell(\cdot)$ is the semilinear form given by

$$\ell(v) = \int_\Sigma \overline{v} \mathcal{L}\left(u^{inc} \right) ds.$$

Two Circular Scatterers

D_1: centered at $(-7, 1)$ with radius 2

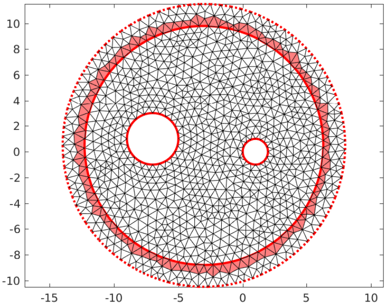
D_2: centered at $(1, 0)$ with radius 1

Wave number $k = \pi$ or $\lambda = 2$

Uniform degree of approximation $p = 8$

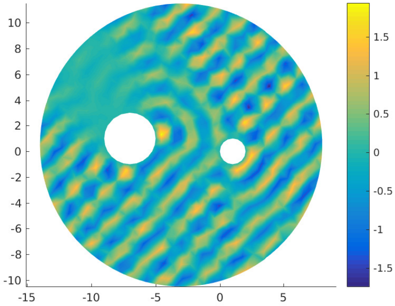
Lagrangian basis functions

Incident direction is $\frac{3}{4}\pi$

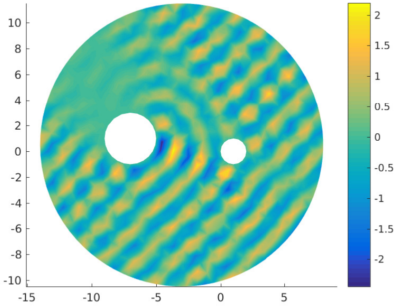


Note that $\|u - u_{hp}\|_W \leq h^p \|u\|_{\Omega,p+1}$.

Numerical Results - Two Plots



(*a*) Real part total field



(*b*) Imaginary part total field

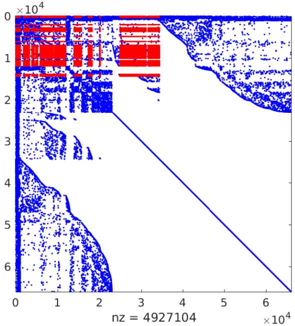
Two circular scatterers

Number of triangles: 2,046

Total Dofs: 65,991

Nonzero entries: 8,797,387 (0.2 %)

Symmetric and nonsymmetric entries



$$a(u,v) = \int_\Omega \overline{\nabla v} \cdot \mathcal{A} \nabla u dA - k^2 \int_\Omega \overline{v} n u dA$$

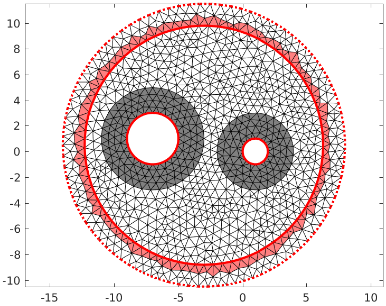
$$- \int_\Sigma \overline{v} \mathcal{L}\left(I^R[u,\Phi]\right) ds - i\lambda \int_\Sigma \overline{v} u ds$$

Comments - References

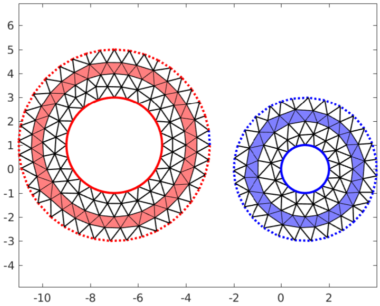
- Overlapping Solution, C. Hazard and M. Lenoir, SIAM J. Math. Anal., 1996
- Existence Uniqueness for FEM & convergence analysis ($p = 1$) - J.C. and P. Monk, SIAM J. Numer. Anal., 2000
- FEM convergence analysis $p \geq 1$ based, in part, on interpolants related to R and Φ - J.C. Appl. Numer. Math., 2012
- FEM Error Analysis for the Maxwell system - G. Hsiao, P. Monk and N. Nigam - SIAM J. Numer. Anal., 2003

Multiple Domain (Disjoint) Formulation

The aim is to proceed by truncating the unbounded domain $\widehat{\Omega}$ *locally*.



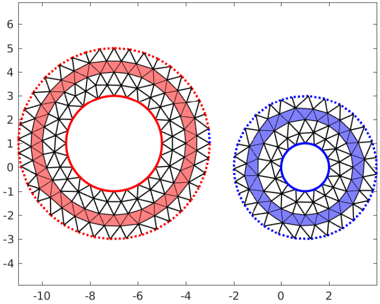
Original mesh



Σ_1 - - - and Σ_2 - - -

Ω_{F_1} and Ω_{F_2}

Denote $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ as the unbounded regions outside of Σ_1 and Σ_2, respectively.



Σ_1 - - - and Σ_2 - - -

Ω_{F_1} and Ω_{F_2}

Multiple Domain Formulation

Denote $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ as the unbounded regions outside of Σ_1 and Σ_2, respectively.

Following Grote and Kirsch*, decompose the scattered field u^{sca} inside $\widehat{\Omega}_1 \cap \widehat{\Omega}_2$ into two outgoing waves u_i^{sca} for $i = 1, 2$ each satisfying

$$\triangle u_i^{sca} + k^2 u_i^{sca} = 0 \quad \text{in} \quad \widehat{\Omega}_i, \tag{8}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u_i^{sca}}{\partial r} - iku_i^{sca} \right) = 0. \tag{9}$$

∗Marcus J. Grote and Christoph Kirsch, *Nonreflecting boundary condiditon for time-dependent multiple scattering* , Journal of Computational Physics, 2007

Multiple Domain Formulation

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$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u_i^{sca}}{\partial r} - iku_i^{sca}\right) = 0. \tag{9}$$

For $\boldsymbol{x}$ in $\widehat{\Omega}_i$

$$\begin{aligned} u_i^{sca}(\boldsymbol{x}) &= \int_{\Omega_{F_i}} \nabla_{\boldsymbol{y}} u_i^{sca}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} R_i(\boldsymbol{y})\Phi(\boldsymbol{x},\boldsymbol{y})\, dA_{\boldsymbol{y}} - k^2 \int_{\Omega_{F_i}} R_i(\boldsymbol{y})\Phi(\boldsymbol{x},\boldsymbol{y})\, u_i^s(\boldsymbol{y}) dA_{\boldsymbol{y}} \\ &- \int_{F_i} u_i^s(\boldsymbol{y}) \frac{\partial \Phi}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\, ds_{\boldsymbol{y}} := I^{R_i}[F_i; u_i^s, \Phi](\boldsymbol{x}) \end{aligned}$$

u_i^{sca} is determined by its values on $\Omega_{F_i} \cup F$.

Multiple Domain Formulation

Denote $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ as the unbounded regions outside of Σ_1 and Σ_2, respectively.

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$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u_i^{sca}}{\partial r} - iku_i^{sca} \right) = 0. \tag{9}$$

For $\boldsymbol{x}$ in $\widehat{\Omega}_i$, $u_i^{sca}(\boldsymbol{x}) := I^R[F_i; u_i^s, \Phi](\boldsymbol{x})$.

The idea is then to *couple* u^{sca} with u_1^{sca} and u_2^{sca} by requiring that

$$u^{sca} = u_1^{sca} + u_2^{sca}$$

on $\Sigma = \Sigma_1 \cup \Sigma_2$.

A new set of equations related to the reduced problem is to find $u \in W$ such that

$$\begin{aligned} \nabla \cdot \mathcal{A}\nabla u + k^2 n u &= 0 && \text{in } \Omega_i, && (10)\\ u &= 0 && \text{on } \Gamma, && (11)\\ \mathcal{L}_i\left(u - I^{R_i}[F_i; u, \Phi] - I^R[F_j; u, \Phi]\right) &= \mathcal{L}_i\left(u^{inc}\right) && \text{on } \Sigma_i && (12) \end{aligned}$$

where $\mathcal{L}_i$ corresponds to the operator $\left(\dfrac{\partial}{\partial \boldsymbol{n}_i} - i\lambda\right)\Big|_{\Sigma_i}$ for $i = 1, 2$.

Consequently,

$$\begin{aligned} \left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\Sigma_i} &= \mathcal{L}_i(u) + i\lambda u \\ &= \mathcal{L}_i(u^{inc}) + \mathcal{L}_i(u^{sca}) + i\lambda u \\ &= \mathcal{L}_i(u^{inc}) + \mathcal{L}_i\left(I^R[F_1; u, \Phi] + I^R[F_2; u, \Phi]\right) + i\lambda u. \end{aligned}$$

The weak form would then be

$$
\begin{aligned}
a(u,v): \quad &= \int_{\Omega_1 \cup \Omega_2} \overline{\nabla v} \cdot \mathcal{A} \nabla u dA - \; k^2 \int_{\Omega_1 \cup \Omega_2} \overline{v} n u dA - i\lambda \int_{\Sigma_1 \cup \Sigma_2} \overline{v} u ds \\
&\quad - \int_{\Sigma_1} \overline{v} \mathcal{L}_1 \left(I^R[F_1; u, \Phi] + I^R[F_2; u, \Phi] \right) ds \\
&\quad - \int_{\Sigma_2} \overline{v} \mathcal{L}_2 \left(I^R[F_1; u, \Phi] + I^R[F_2; u, \Phi] \right) ds \\
&= \int_{\Sigma_1} \overline{v} \mathcal{L}_1 \left(u^{inc} \right) ds + \int_{\Sigma_2} \overline{v} \mathcal{L}_2 \left(u^{inc} \right) ds \\
&:= \ell(u^{inc}).
\end{aligned}
$$

Two circular scatterers

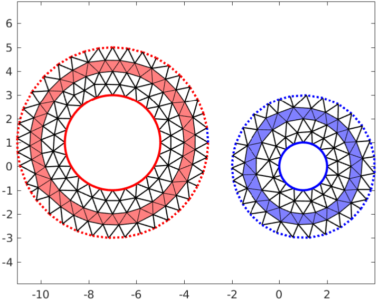
D_1: centered at $(-7, 1)$ with radius 2

D_2: centered at $(1, 0)$ with radius 1

Wave number $k = \pi$ or $\lambda = 2$

Uniform degree of approximation $p = 8$

Incident direction is $\frac{3}{4}\pi$



Two circular scatterers

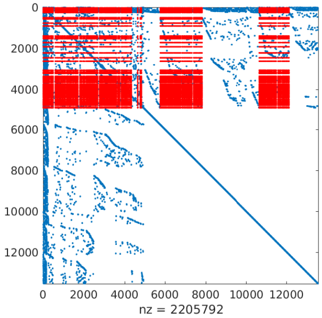
Number of triangles: 410

Total Dofs: 13,552

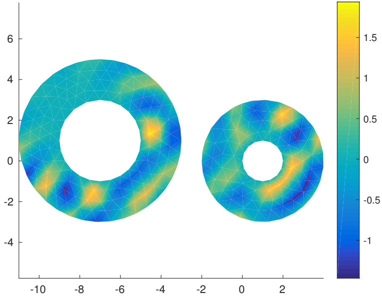
Nonzero entries: 2,989,744 (1.6 %)

Symmetric and nonsymmetric entries

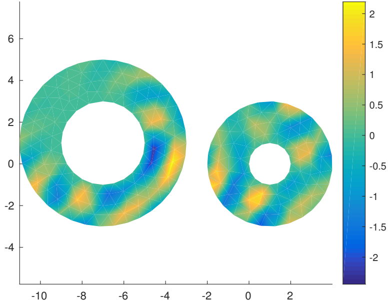
$$a(u,v) = \int_{\Omega_1 \cup \Omega_2} \overline{\nabla v} \cdot \mathcal{A} \nabla u dA - k^2 \int_{\Omega_1 \cup \Omega_2} \overline{v} n u dA$$
$$- \int_{\Sigma_1} \overline{v} \mathcal{L}_1 \left(I^R[F_1; u, \Phi] + I^R[F_2; u, \Phi] \right) ds$$
$$- \int_{\Sigma_2} \overline{v} \mathcal{L}_2 \left(I^R[F_1; u, \Phi] + I^R[F_2; u, \Phi] \right) ds$$
$$- i\lambda \int_{\Sigma_1 \cup \Sigma_2} \overline{v} u ds$$



System Matrix



(a) Real part total field



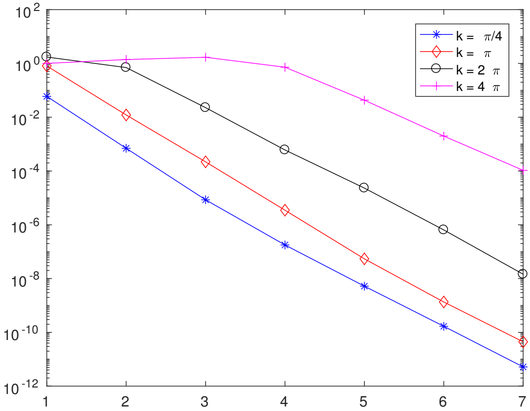
(b) Imaginary part total field

$$\|\boldsymbol{x}_{\text{one mesh}} - \boldsymbol{x}_{\text{two meshes}}\|_F = 2.2117e - 07$$

Degree	Num. Dofs	Num. Nonzero	Percent Nonzero
1	259	8894	13.3
2	928	54,544	6.3
3	2,007	16,7073	4.1
4	3,496	379,064	3.1
5	5,395	725,560	2.5
6	7,704	1,244,064	2.1
7	10,423	1,974,539	1.8
8	13,552	2,989,744	1.6

H^1-norm of the difference between the computed solution and the *true* solution versus the uniform order of approximation.

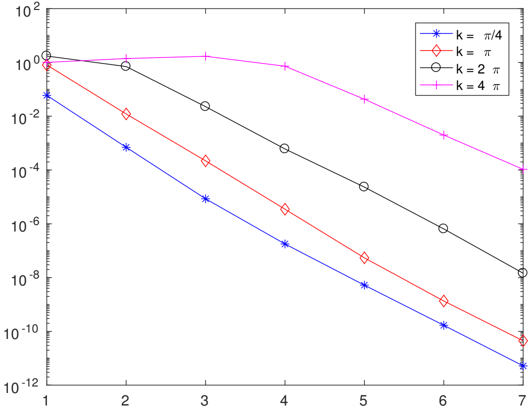
- Degree of approx ranges from $p = 1$ to $p = 7$



Preliminary Convergence Results

H^1-norm of the difference between the computed solution and the *true* solution versus the uniform order of approximation.

- Degree of approx ranges from $p = 1$ to $p = 7$
- The so-called *true* solution is the solution computed on the single mesh with $p = 8$.



$$
\begin{aligned}
a(u,v) &= \sum_i \left(\int_{\Omega_i} \overline{\nabla v} \cdot \mathcal{A} \nabla u dA - k^2 \int_{\Omega_i} \overline{v} n u dA - i\lambda \int_{\Sigma_i} \overline{v} u ds \right) \\
&- \sum_i \left(\int_{\Sigma_i} \overline{v} \mathcal{L}_i \left(I^R[F_i; u, \Phi] \right) ds \right) - \sum_{i,j;i\neq j} \int_{\Sigma_i} \overline{v} \mathcal{L}_i \left(I^R[F_j; u, \Phi] \right) ds \\
&= \sum_i \int_{\Sigma_i} \overline{v} \mathcal{L}_i \left(u^{inc} \right) ds \\
&= \ell(u).
\end{aligned}
$$

We set

$$
\begin{aligned}
[\mathrm{M}]_{m,n} &= \sum_i \int_{\Omega_i} \overline{\nabla v_m} \cdot \mathcal{A} \nabla u_n dA, \\
[\mathrm{G}]_{m,n} &= \sum_i \int_{\Omega_i} \overline{v}_m n u_n dA, \\
[\mathrm{S}]_{m,n} &= \sum_i \int_{\Sigma_i} \overline{v}_m u_n ds,
\end{aligned}
$$

It should be noted that M, G and S are block diagonal. That is to say, if denote $\mathrm{M}^{(i)}$, $\mathrm{G}^{(i)}$ and $\mathrm{S}^{(i)}$ as the parts of M, G over Ω_i and S over Σ_i,, then

$$
\mathrm{M} = \mathrm{blockdiag}[\mathrm{M}^{(1)}, \mathrm{M}^{(2)}, ..., \mathrm{M}^{(n)}]
$$

with similar forms for G and S.

The remaining term contains the (artificial) boundary conditions on the individual Ω_i's

$$[\mathsf{E}^{(i)}]_{m,n} = \sum_i \int_{\Sigma_i} \overline{v}_m \mathcal{L}_i \left(I^R[F_i; u_n, \Phi] \right) ds$$

as well as the nonlocal coupling due to the representation of the outgoing fields

$$[\mathsf{C}^{(i,j)}]_{m,n} = \sum_i \int_{\Sigma_i} \overline{v}_m \mathcal{L}_i \left(I^R[F_j; u_n, \Phi] \right) ds.$$

The system matrix is then written as

$$\mathsf{A} = \mathsf{G} - k^2 \mathsf{M} - i\lambda \mathsf{S} - \mathsf{B}$$

where

$$\mathsf{B} = \begin{bmatrix} \mathsf{E}^{(1)} & \mathsf{C}^{(1,2)} & \dots & \mathsf{C}^{(1,n)} \\ \mathsf{C}^{(2,1)} & \mathsf{E}^{(2)} & \dots & \mathsf{C}^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{C}^{(n,1)} & \mathsf{C}^{(n,2)} & \dots & \mathsf{E}^{(n)} \end{bmatrix}.$$

The entire system is then given by

$$\mathrm{A}\boldsymbol{x} = \mathrm{L}$$

where L is the column vector

$$\mathrm{L} = \begin{bmatrix} \mathrm{L}^{(1)} \\ \mathrm{L}^{(2)} \\ \vdots \\ \mathrm{L}^{(n)} \end{bmatrix}$$

defined as

$$[\mathrm{L}^{(i)}]_n = \int_{\Sigma_i} \overline{v}_n \mathcal{L}_i \left(u^{inc}\right) ds.$$

This is the straightforward *all in one solve*.

Iteration Scheme

1. Determine an initial solution to the individual problems

$$(\mathsf{G}^{(i)} - k^2\mathsf{M}^{(i)} - i\lambda\mathsf{S}^{(i)} - \mathsf{B}^{(i)} - \mathsf{E}^{(i)})\boldsymbol{x}_0^{(i)} = \mathsf{L}^{(i)}$$

2. Then, *until some criteria* solve for $i = 1, ..., n$

$$(\mathsf{G}^{(i)} - k^2\mathsf{M}^{(i)} - i\lambda\mathsf{S}^{(i)} - \mathsf{B}^{(i)} - \mathsf{E}^{(i)})\boldsymbol{x}_j^{(i)} = \mathsf{L}^{(i)} + \sum_{i,j,i\neq j} \mathsf{C}^{(i,j)}\boldsymbol{x}_{j-1}^{(i)}.$$

Two circular scatterers

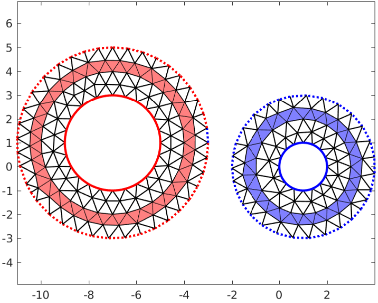
Once we have $x_0^{(1)}$ and $x_0^{(2)}$, we solve

$$\mathsf{A}^{(1)}\boldsymbol{x}_j^{(1)} = \mathsf{L}^{(1)} + \mathsf{C}^{(1,2)}\boldsymbol{x}_{j-1}^{(2)}$$

and then

$$\mathsf{A}^{(2)}\boldsymbol{x}_j^{(1)} = \mathsf{L}^{(2)} + \mathsf{C}^{(2,1)}\boldsymbol{x}_{j-1}^{(1)}$$

for $j = 1, \dots$



Two meshes

Two circular scatterers

Once we have $x_0^{(1)}$ and $x_0^{(2)}$, we solve

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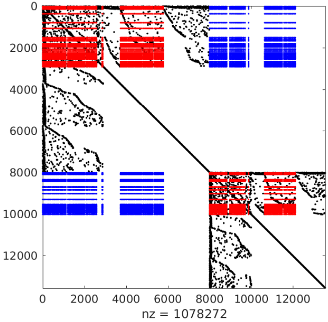
and then

$$\mathrm{A}^{(2)}\boldsymbol{x}_j^{(1)} = \mathrm{L}^{(2)} + \mathrm{C}^{(2,1)}\boldsymbol{x}_{j-1}^{(1)}$$

for $j = 1, \ldots$
Note:

$$\mathrm{A}^{(i)} = \mathrm{G}^{(i)} - k^2\mathrm{M}^{(i)} - i\lambda\mathrm{S}^{(i)} - \mathrm{B}^{(i)} - \mathrm{E}^{(i)}.$$



Two meshes

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Once we have $x_0^{(1)}$ and $x_0^{(2)}$, we solve

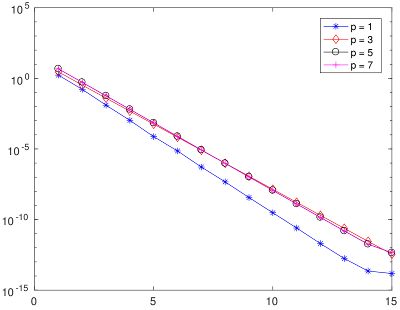
$$\mathrm{A}^{(1)}\boldsymbol{x}_j^{(1)} = \mathrm{L}^{(1)} + \mathrm{C}^{(1,2)}\boldsymbol{x}_{j-1}^{(2)}$$

and then

$$\mathrm{A}^{(2)}\boldsymbol{x}_j^{(1)} = \mathrm{L}^{(2)} + \mathrm{C}^{(2,1)}\boldsymbol{x}_{j-1}^{(1)}$$

for $j = 1, \ldots$

$$\left\| x_j - x_{true} \right\|_F$$



Using $j = 1, \ldots, 15$ with $k = 2\pi$

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