First-Order Logic

- First-Order Logic, FOL, FOPC
- Definition of entailment.
- Inference Methods
- Formal language, well understood
- Not the only possibility

Language

1. Syntax
2. Semantics
3. Pragmatics

Logical Symbols

Punctuation "(", ")", "."

Connectives **Negation** $\neg$

- **Conjunction** $\wedge$
- **Disjunction** $\vee$
- **Quantifiers** $\forall, \exists$
- **Equality** $=$

Variables , an infinite supply of symbols

$x, x_1, x_2, \ldots$

Non-Logical Symbols

Predicate Symbols An infinite supply of symbols for each arity.

DOG

OLDERTHAN

Function Symbols An infinite supply of symbols for each arity.

BESTFRIEND

CHARLIEPACK

Terms

The set of terms of FOL is the least set satisfying these conditions.

1. Every variable is a term.
2. If $t_1, t_2, \ldots, t_n$ are terms and f is a function of arity n, then $f(t_1, t_2, \ldots, t_n)$ is a term.

Formulas

The set of formulas of FOL is the least set satisfying these conditions.

1. If $t_1, t_2, \ldots, t_n$ are terms and P is a predicate symbol of arity n, then $\mathrm{P}(t_1, t_2, \ldots, t_n)$ is a formula.

2. If t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.

3. If α and β are formulas, and x is a variable, then $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $\forall x.\alpha$, and $\exists x.\alpha$ are formulas.

atomic formulas, atoms well formed formulasm, wffs.

Propositional Subset

- All predicates of arity 0.
- No terms (no function symbols, no variables).
- No quantifiers.

Abbreviations

implication $\rightarrow$, $\supset$

equivalence $\equiv$

Terminology

- Bound Variable
- Free Variable
- Sentence

Interpretations

An interpretation $\Im$ in FOL is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$

- $\mathcal{D}$ is called the domain of interpretation. It is any non-empty set of objects.
- $\mathcal{I}$ is the interpretation mapping from the non-logical symbols to functions and relations over $\mathcal{D}$.

Interpretation Mapping

$$\mathcal{I}[\mathrm{P}] \subseteq \mathcal{D} \times \ldots \mathcal{D}$$

$$\mathcal{I}[\mathrm{F}] \subseteq \mathcal{D} \times \ldots \mathcal{D} \rightarrow \mathcal{D}$$

Denotation

variable assignment μ over $\mathcal{D}$

1. If x is a variable then $[\![x]\!]_{\Im,\mu} = \mu[x]$
2. If $t_1, \ldots, t_n$ are terms, and $\textsc{f}$ is a function symbol of arity n then

$$[\![\textsc{f}(t_1, \ldots t_n)]\!]_{\Im,\mu} = \mathcal{I}(\textsc{f})([\![t_i]\!]_{\Im,\mu}, \ldots, [\![t_i]\!]_{\Im,\mu})$$

Satisfaction

1. $\Im, \mu \models P(t_1 \ldots t_n)$ iff $\langle d_1, \ldots, d_n \rangle \in \mathcal{I}(\mathrm{P})$, where $d_i = [\![t_i]\!]_{\Im,\mu}$
2. $\Im, \mu \models t_1 = t_2$ iff $[\![t_1]\!]_{\Im,\mu}$ and $[\![t_2]\!]_{\Im,\mu}$ and are the same element of D.
3. $\Im, \mu \models \neg\alpha$ iff it is not the case that $\Im, \mu \models \alpha$
4. $\Im, \mu \models (\alpha \wedge \beta)$ iff $\Im, \mu \models \alpha$ and $\Im, \mu \models \beta$.
5. $\Im, \mu \models (\alpha \vee \beta)$ iff $\Im, \mu \models \alpha$ or $\Im, \mu \models \beta$.
6. $\Im, \mu \models \exists x.\alpha$ iff $\Im, \mu' \models \alpha$ for some variable assignment μ' that differs from μ on at most x.
7. $\Im, \mu \models \forall x.\alpha$ iff $\Im, \mu' \models \alpha$ for every variable assignment μ' that differs from μ on at most x.

Important Concepts

- Logical Consequence, Logical Entailment
- Satisfiable
- Unsatisfiable
- Sound
- Complete