Constraint Satisfaction Problems

Chapter 5

Section 1 – 3

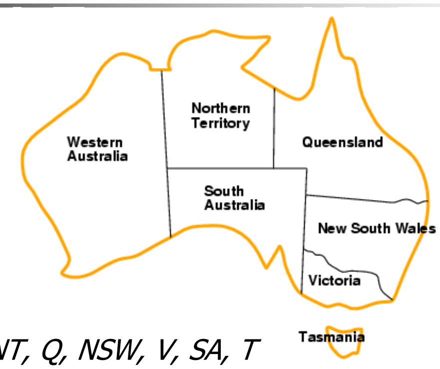
Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

Constraint satisfaction problems (CSPs)

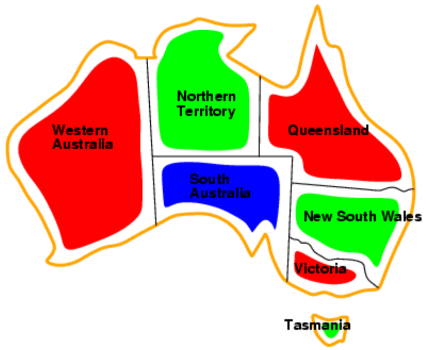
- Standard search problem:
  - state is a "black box" – any data structure that supports successor function, heuristic function, and goal test
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring



- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}

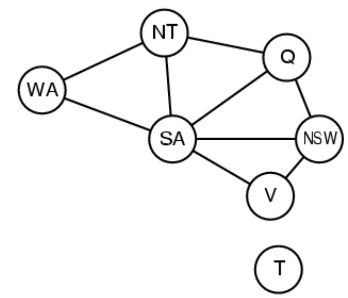
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green□

Constraint graph

- Binary CSP: each constraint relates two variables□
- Constraint graph: nodes are variables, arcs are constraints□



Varieties of CSPs

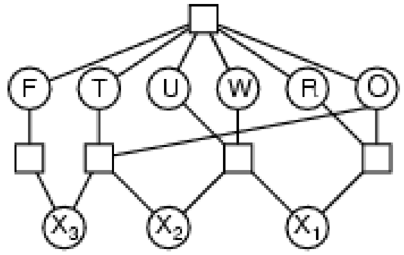
- Discrete variables□
  - finite domains:
    - *n* variables, domain size *d* → $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- Continuous variables□
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green□
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA□
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints□

Example: Cryptarithmetic

```
  T W O
+ T W O
-------
F O U R
```



- Variables: *F T U W R O* X_1 X_2 X_3
- Domains: {*0,1,2,3,4,5,6,7,8,9*}
- Constraints: *Alldiff (F,T,U,W,R,O)*□
  - $O + O = R + 10 \cdot X_1$□
  - $X_1 + W + W = U + 10 \cdot X_2$□
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F$, $T \neq 0$, $F \neq 0$□

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class□
- Timetabling problems□
  - e.g., which class is offered when and where?□
- Transportation scheduling□
- Factory scheduling□

- Notice that many real-world problems involve real-valued variables□

Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it□

States are defined by the values assigned so far□

- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
  → fail if no legal assignments□
- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   → use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. b = (n - *l*)d at depth *l*, hence n! · d^n leaves□

Backtracking search

- Variable assignments are commutative}, i.e.,
[WA = red then NT = green] same as [NT = green then WA = red]□

- Only need to consider assignments to a single variable at each node
  → b = d and there are d^n leaves□

- Depth-first search for CSPs with single-variable assignments is called backtracking search□

- Backtracking search is the basic uninformed algorithm for CSPs□

- Can solve *n*-queens for $n \approx 25$□

Backtracking search

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)
function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```


Backtracking example

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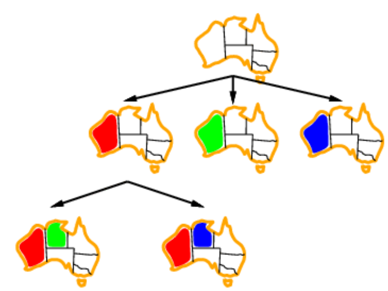
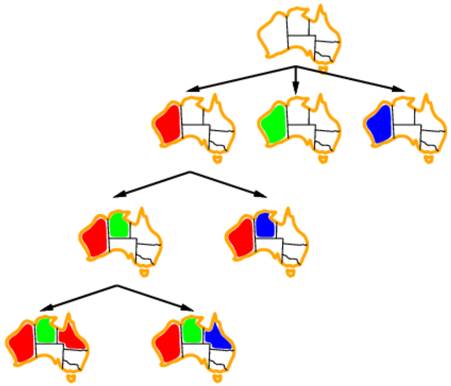


Backtracking example

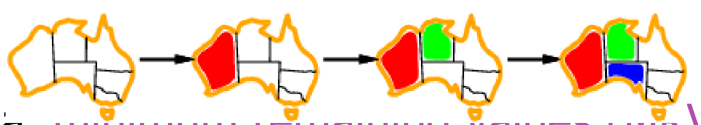


Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:□
  - Which variable should be assigned next?□
  - In what order should its values be tried?□
  - Can we detect inevitable failure early?□

Most constrained variable

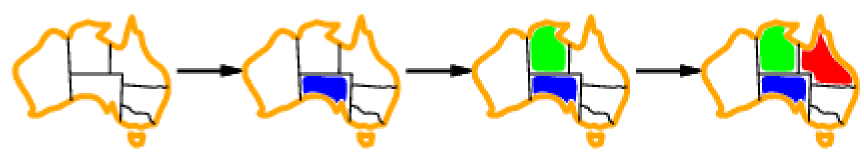
- Most constrained variable:
  choose the variable with the fewest legal values□



- a.k.a. minimum remaining values (MRV) heuristic□

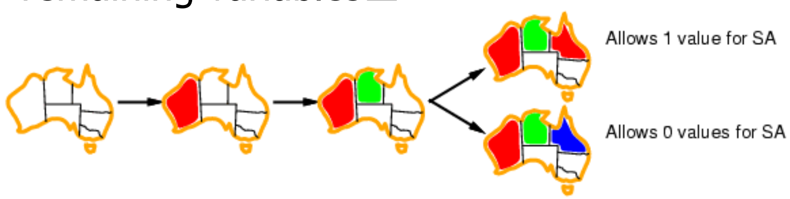
Most constraining variable

- Tie-breaker among most constrained variables
- Most constraining variable:□
  - choose the variable with the most constraints on remaining variables□



Least constraining value

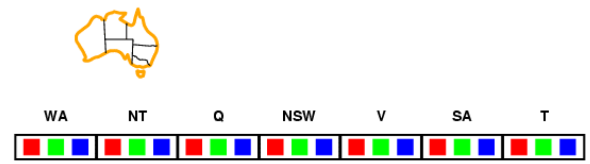
- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



- Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



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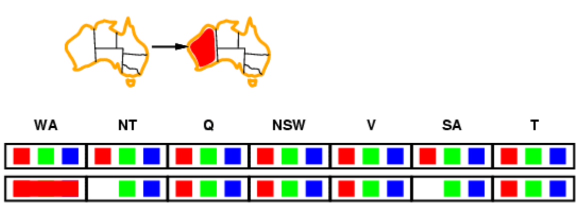

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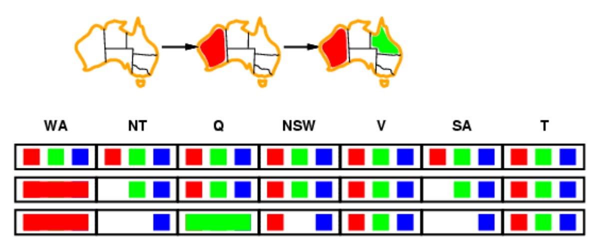

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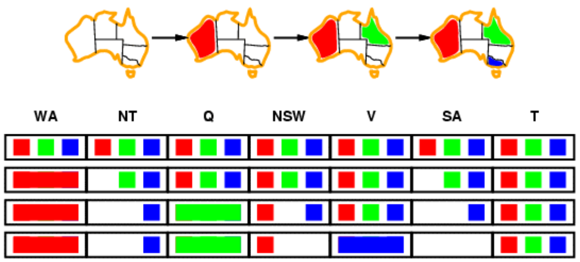
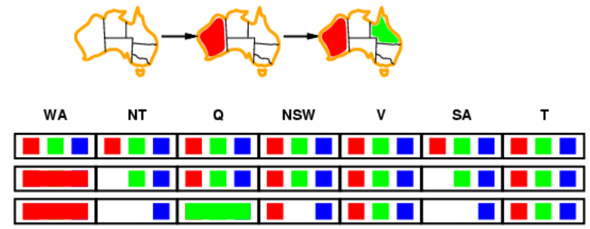


Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

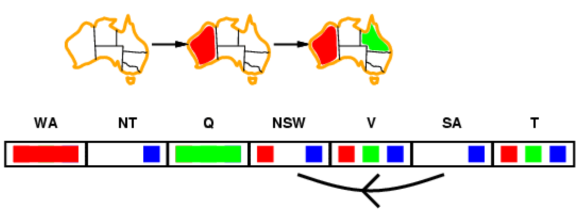


- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* → *Y* is consistent iff

  for every value *x* of *X* there is some allowed *y*



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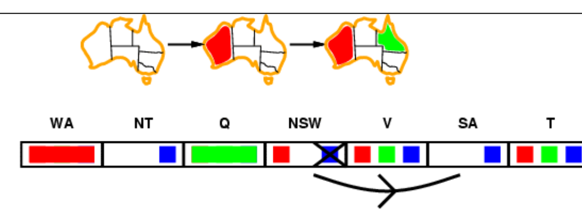
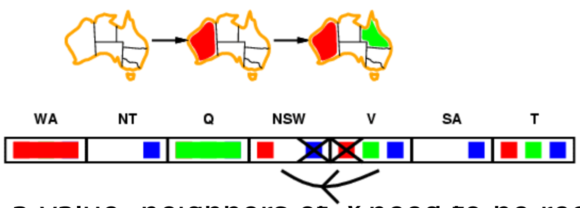


Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff□

  for every value x of X there is some allowed y□



- If X loses a value, neighbors of X need to be rechecked□

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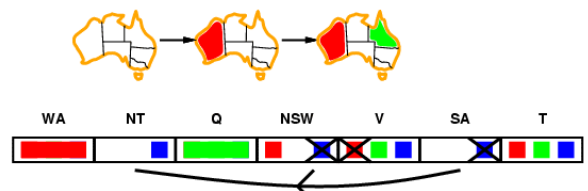


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking□
- Can be run as a preprocessor or after each assignment□

□

Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if RM-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue

function RM-INCONSISTENT-VALUES( Xi, Xj) returns true iff remove a value
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy constraint(Xi, Xj)
         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

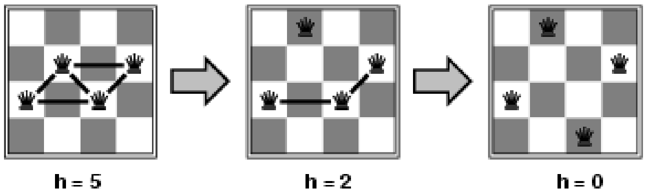
- Time complexity: $O(n^2 d^3)$□

Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned□
- To apply to CSPs:□
  - allow states with unsatisfied constraints□
  - operators reassign variable values□
- Variable selection: randomly select any conflicted variable□
- Value selection by min-conflicts heuristic:□
  - choose value that violates the fewest constraints□
  - i.e., hill-climb with *h(n)* = total number of violated constraints□

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of attacks



- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice