The Mathematics of Internet Search Engines

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Outline

Mathematics of Information Retrieval

Information retrieval methods such as

- traditional vector space methods (keyword searches)
- Google's PageRank
- HITS (Ask.com)
- SALSA

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- aspects which are accessible early in the curriculum
- other aspects which provide appropriate investigations at intermediate and advanced levels

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- Nothing to guarantee the quality of a site.
- Very susceptible to manipulation (how?).

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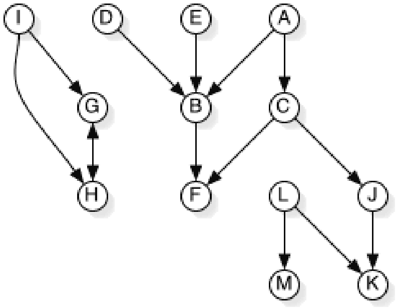
- The web consists of two sets: its pages *and* its links.
- Page-Brin-Kleinberg: use the organic and social link structure of the web in order to *rank* pages.
- The web is modeled as a *directed graph*

Directed Graphs

A *directed graph* is a collection of nodes (the web pages) together with a collection of arrows pointing from one node to another (the links).

Example



Application for the Classroom - 1

Graph theory has become a common topic introduced in

- mathematics survey courses for non-science, non-engineering majors.
- discrete mathematics courses.

Typical focus is on routing and scheduling problems:

The Web provides another very relevent, useful model utilizing directed graphs.

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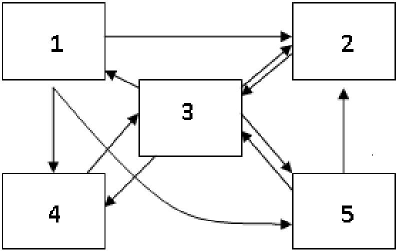
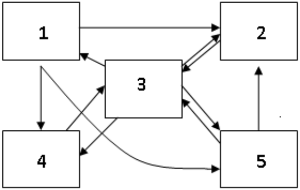
- search engine page rankings

The Mathematics of Google's PageRank



Each page is assigned a *rank*, which is a numerical value between 0 and 1. Then, when a keyword search is performed, results are returned according to their ranks, with higher ranks returned first.

PageRank Example



Each page inherits its rank from those sites linking to it. So, for example, Page 2 gets:

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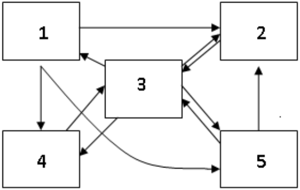

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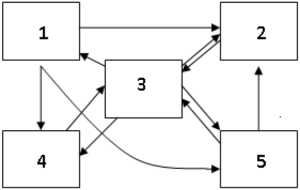

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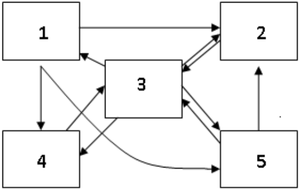

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Letting r_i denote the rank of page i, $i = 1, \ldots, 5$, we obtain the following system of linear equations:

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0r_1 + 1r_2 + 0r_3 + 1r_4 + 1/2r_5 &= r_3 \\
1/3r_1 + 0r_2 + 1/4r_3 + 0r_4 + 0r_5 &= r_4 \\
1/3r_1 + 0r_2 + 1/4r_3 + 0r_4 + 0r_5 &= r_5
\end{aligned}
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Application for the Classroom - 2

Solutions to systems of linear equations are often covered in

Google's PageRank algorithm provides a relevent and modern example of a problem whose solution is given by such a system.

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- too big to solve by hand
- good opportunity to discuss the role of technology

PageRank Example

We may rewrite the previous system of linear equations in matrix form.

$$\begin{bmatrix} 0 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/4 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 1/2 \\ 1/3 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/4 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}$$

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In this example, the system has a 1-parameter family of solutions. There is a unique solution whose entries are positive and sum to 1. This vector is called the *pagerank vector*, and is given by

$$\begin{bmatrix} .103 & .207 & .414 & .138 & .138 \end{bmatrix}$$

Application for the Classroom - 3

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Linear Algebra

- Writing the system as $Hx = x$, we clearly have an eigenvector problem.
- At this point, we encounter several "why"'s.
- But at a minimum, the PageRank algorithm gives a new example of a problem whose solution is provided by an eigenvector of a matrix.

The Hyperlink Matrix

The matrix obtained on the previous slide is called the *hyperlink matrix* of the web:

$$H = \begin{bmatrix} 0 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/4 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 1/2 \\ 1/3 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/4 & 0 & 0 \end{bmatrix}$$

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- completely describes the link structure of the web.
- The pagerank vector is a particular eigenvector associated to the eigenvalue 1.

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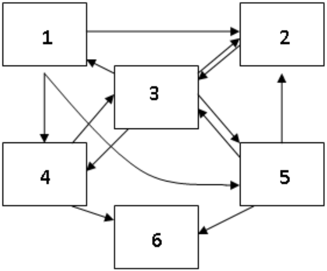

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  - consider not just the action of following links, but also of directly typing in a URL.

The Dangling Node

Google's PageRank algorithm replaces the previous H with a "tweaked" matrix

$$S = \begin{bmatrix} 0 & 0 & 1/4 & 0 & 0 & 1/6 \\ 1/3 & 0 & 1/4 & 0 & 1/3 & 1/6 \\ 0 & 1 & 0 & 1/2 & 1/3 & 1/6 \\ 1/3 & 0 & 1/4 & 0 & 0 & 1/6 \\ 1/3 & 0 & 1/4 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 1/2 & 1/3 & 1/6 \end{bmatrix}$$

where each $1/6$ in the last column represents the probability of randomly visiting one of the 6 pages by directly typing in its URL.

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- G is a positive, column stochastic matrix (this is good).

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- Why does it have a unique relevant solution?

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Let $A > 0$ *with* $r = \rho(A)$. *Then the following are true.*

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- *Note: When A is stochastic, $r = 1$.*

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- The power method provides an iterative technique for computing a dominant eigenpair of G.
- Typically converges in less than 20 iterations.
- Nice example for Numerical Analysis.

Further Reading

1. *Understanding Search Engines*, by Berry and Browne, 2005

   

2. *Google's PageRank and Beyond*, by Langville and Meyer, 2006